Rotated sandpiles: The role of grain reorganization and inertia

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We present a coupled-map lattice model of a sandpile which relaxes via the dual mechanisms of surface flow and bulk reorganization. We focus on the situation where reorganization dominates flow and examine the response of our model to rotation; the event size distribution is analyzed as a function of the model parameters. In regions of phase space where the effects of excess volume and grain inertia are both large, we see a characteristic double peak in the size distribution, indicating a preferred scale for large avalanches. We explain this, and the associated configurational memory, using a simple picture of grain-cluster coupling. We next compare the distribution functions with those produced by random deposition and, in the latter case, present results for the event size distribution function in terms of the size of the deposited grains. Finally, we show the connection of our model to earlier scale-invariant models, and discuss our results. [S1063-651X(96)09306-3]

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INTRODUCTION

Although sandpile models were first introduced by Bak *et al.* [1] as examples of extended dissipative systems they have generated considerable interest in their own right in the context of granular media [2,3] leading to many experimental, numerical, and theoretical investigations [4]. Most numerical investigations have concentrated on simple lattice-based models, such as cellular automata, to explore the different factors influencing granular flow; in particular, discrete sandpile models have been used recently to investigate the impact of granular disorder [5], grain inertia [6], and sliding friction [7] on the statistical distribution of avalanche sizes in driven sandpiles. In this paper we model a sandpile on a rotating base, which is a case of great experimental importance [3], and examine the effect of granular reorganization and inertia on its avalanche spectrum.

The avalanche size distribution has been of great significance in the development of sandpile modeling because experimental results [8,9] are at variance with the original model of Bak *et al.* [1]; while the latter predicts a simple power law for the avalanche size distribution function, indicating scale-invariant dynamics, the former show that in practice, larger avalanches occur more frequently than predicted by the power law describing the distribution of the smaller events. These experimental results demonstrate that one particular range of avalanche sizes is preferred, and have led to a large body of theoretical and numerical work which aims to identify the underlying physical mechanisms for such behavior.

In our search for a suitable explanation we constructed a cellular automaton model of a reorganizing sandpile [5] based on a model [10] that identifies two distinct dynamical mechanisms for granular relaxation: a faster mechanism corresponding to the motion of grains moving independently of their clusters, and a slower mechanism corresponding to the collective reorganization of grains within their clusters. The avalanche spectrum obtained in that work was in agreement with experiment [8,9], and attributed the breakdown of scale invariance to the competition between these two mechanisms.

In this paper we model an experimental situation which forms the basis of many traditional [11] as well as modern [8] experiments; a sandpile in a rotating cylinder. We consider the dynamics of sand in a ''half cylinder'' that is rotating slowly around its axis and we suppose that the sand is uniformly distributed in the direction of the axis. Our model is therefore essentially one-dimensional. The driving force arising from rotation continually effects the stability of the sand at all positions in the pile and is therefore distinct from random deposition. We include both surface flow and internal restructuring [4,5] as mechanisms of sandpile relaxation and focus on a situation where reorganization within the pile dominates the flow. This situation, of a sandpile subjected to slow rotation or tilt, has been formulated elsewhere [4,12] in terms of continuum equations. Finally, we look at the effect of random driving forces in our model and compare the results with those from other models.

MODEL

Since the effect of grain reorganization driven by slow tilt is easiest to visualize from a continuum viewpoint our model incorporates grains which form part of a continuum so that the column heights, h_i, are real variables $h_i \in \mathbb{R}$; the column identities, $1 \leq i \leq L$, are discrete as usual. We consider granular driving forces, f_i, that include, in addition to a term that drives the normal surface flow, a contribution that is proportional to the deviation of the column height from an ''ideal'' height; this ideal height is a simple representation of a natural random packing of the grains in a column, so that columns which are taller (shorter) than ideal would be relatively loosely (closely) packed, and driven to consolidate (dilate) when the sandpile is perturbed externally. Our motivation for this choice comes from previous work [13] where we have shown that a shaken sandpile tends to consolidate or dilate,

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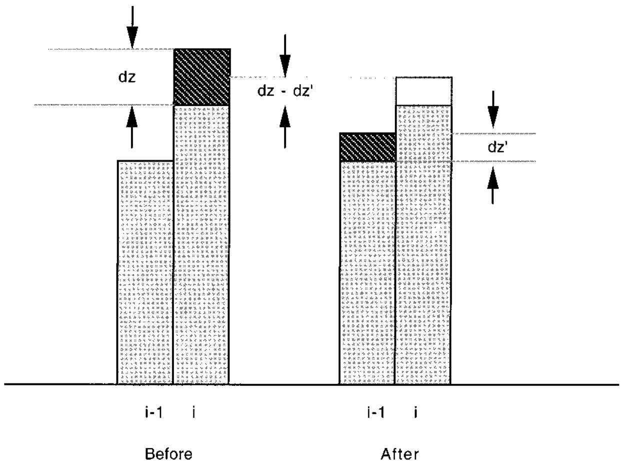

FIG. 1. A schematic diagram showing the column height changes that describe a single relaxation event in the CML sandpile model.

depending on its initial state; we generalize this here to include driving in the form of rotation and deposition. Thus

$$f_i = k_1(h_i - iaS_0) + k_2(h_i - h_{i-1} - aS_0), \quad i \neq 1, \tag{1}$$

where h_i are the column heights, k_1 and k_2 are constants, a is the lattice spacing, and iaS_0 is the ideal height of column i. We note that (a) the first term, which depends on the absolute height of the sandpile, corresponds to a force that drives column compression or expansion towards the ideal height. Since we normally deal with columns which are more dilated than their normal height, we will henceforth talk principally about column compression; (b) the second term is the usual term driving surface flow, which depends on local slope, or height differences; the offset of S_0 is the ideal slope from which differences are measured.

Equation (1) suggests the redefinition of heights $az_i \equiv h_i - iaS_0$, which leads to the dimensionless representation:

$$f_i = (k_1/k_2)z_i + (z_i - z_{i-1}), \quad i \neq 1. \tag{2}$$

When column i is subject to a force greater than or equal to the threshold force f_{th}, the height changes are as follows:

$$\begin{aligned} z_i &\to z_i - \delta z, \\ z_{i-1} &\to z_{i-1} + \delta z', \quad i \neq 1. \end{aligned} \tag{3}$$

The column-height changes that correspond to a typical relaxation event are illustrated in Fig. 1. Thus, the height δz removed from column i, due to a local driving force that exceeds the threshold force, leads to a flow of grains, with total height increment $\delta z'$, from column i onto column $i-1$, and a consolidation of the grains in column i which reduces the column height by $(\delta z - \delta z')$. This reorganization clearly expresses the action of two relaxation mechanisms. The decomposition of the relaxation, that is, a particular choice for δz and $\delta z'$, is discussed below in analogy with a previously established model for earthquake behavior.

The coupling between the column heights expressed in (3) may lead to the propagation of instabilities along the sandpile and hence to avalanches. Avalanches have also been discussed widely in the context of earthquakes [14], and one aim of this paper is to draw analogies between sandpiles and earthquakes. We choose a discrete model of earthquakes, put forward by Nakanishi [15], to highlight those features which are common to our sandpile model and earthquake models and we follow the notation of [15] wherever possible.

We choose the force relaxation function, $f_i - f_i'$ to be [15]

$$f_i - f_i' = f_i - f_{\text{th}}\left(\frac{(2-\delta f)^2/\alpha}{(f_i - f_{\text{th}})/f_{\text{th}} + (2-\delta f)/\alpha} - 1\right), \tag{4}$$

where f_i and f_i' are the granular driving forces on column i before and after a relaxation event. This function has a minimum value ($=\delta f \; f_{\text{th}}$) when $f_i = f_{\text{th}}$, and increases monotonically with increasing f_i; this form models the stick-slip friction associated with sandpiles and earthquakes. For driving forces, f_i, below the threshold force f_{th} nothing happens but, for forces that exceed this threshold, the size of relaxation events increases in proportion to the excess force. Accordingly, the minimum value of the function (4) is known as the minimum event size and its initial rate of increase, $\alpha = d(f_i - f_i')/d(f_i - f_{\text{th}})$ at $f_i = f_{\text{th}}$, is called the amplification [15]. In our sandpile model amplification refers to the phenomenon whereby grains collide with each other during an avalanche so that their inertial motion contributes to the buildup of the avalanche; thus α is an expression of granular inertia [6].

Using Eq. (2) we can rewrite the map in terms of the driving forces as (see Appendix A):

$$\begin{aligned} f_i - f_i' &= 2\,\delta z/\Delta, \\ \delta z' &= \delta z/(1 + k_1/k_2), \end{aligned} \tag{5}$$

$$f_{i-1}' - f_i = f_{i+1}' - f_{i+1} = -\Delta(f_i' - f_i)/2, \quad i \neq 1 \text{ or } L. \tag{6}$$

In both sandpile and earthquake models the amount of redistributed force at a relaxation event is governed by the parameter $\Delta = 2(1 + k_1/k_2)/[1 + (1 + k_1/k_2)^2]$; since the undistributed force is ''dissipated'' $(1-\Delta)$ becomes the dissipation coefficient (Appendix A). Note, however, that in our sandpile model, this dissipation is linked to nonconservation of the sandpile *volume* arising from the compression of columns towards their ideal heights and is therefore linked to the phenomenon of internal granular reorganization.

We use boundary conditions appropriate to a sandpile in a rotating cylinder; open at $i=0$ and closed at $i=L$. Equations (4) and (6) give a prescription for the evolution of forces, $\{f_i\}$ $i=1,L$, so that any forces in excess of the threshold force are relaxed according to (6) and redistributed according to (4). Alternatively this sequence of events can be followed in terms of the redistribution of column heights according to (3) and (5). The prescription, (4), (6), classifies the model as a local and unlimited sandpile in the framework given by Kadanoff *et al.* [16].

We will show below that for all $\Delta \neq 1$, the largest part of the volume change during relaxation occurs as a result of consolidation; the quantity of interest is thus the difference between the old and new configurations, rather than the mass exiting the sandpile [5]. A measure of this change is the quantity $\ln M = \ln \Sigma_i (f_i - f_i') = \ln[\Sigma_i (k_1/k_2)(z_i - z_i') + z_L - z_L']$, where z_i' is the height of column i immediately after a relaxation event; this quantity is the analogue of the event magnitude in earthquake models [14,15]. We will discuss the variation of this quantity as a function of model parameters in the next section; in particular, we will compare the response of our rotated sandpile model to that of the same model subjected to random deposition.

RESULTS

Rotated sandpile

For a sandpile in a rotating cylinder, the tilting of the sandpile results in changes of slope over the complete surface in a continuous manner (in contrast to the case of random deposition where the slopes change locally and discontinuously at a deposition event [5]). Our coupled map lattice (CML) model is driven continuously; from a configuration in which all forces f_i are less than the threshold force, elements of height, z_i^+, are added onto each column with

$$z_i^+ = i(f_{\rm th} - f_j)/(1 + jk_1/k_2), \quad i = 1, L, \tag{7}$$

where $f_j = \max(f_i)$. This transformation describes the effect of rotating the base of the sandpile with a constant angular speed until a threshold force arises at column j. (We note that this is distinct from the external driving force in the earthquake model [15], which would correspond, in a sandpile model, to the uniform addition of height elements across the surface.) The response to the tilting is, as described above, a flow of particles down the slope as well as reorganisation of particles within the sandpile.

The predominant effect of our model is to cause volume changes by consolidation, rather than to generate surface flow. Using the relation between force and column height, (2), and integrating from the left, we can construct the shape of a critical sandpile which has driving forces equal to the threshold force on all of its columns; in terms of the variable $\zeta \equiv (1 + \sqrt{1 - \Delta^2})/\Delta$, the critical sandpile has column heights z_i^c given by

$$z_i^c = f_{\rm th}\{1 - \zeta^{-1} \exp[(1 - \zeta)(i - 1)a]\}/(\zeta - 1), \quad \Delta < 1. \tag{8}$$

This shows that, for all $\Delta < 1$, the critical sandpile starts, at $i = 1$, with a slope greater than S_0 and subsequently the slope decreases until it becomes steady, at S_0 for $i \gg 1$ where the constant deviation of the column heights from their ideal values is given by $f_{\rm th}\Delta/(1 - \Delta + \sqrt{1 - \Delta^2})$ (Fig. 2). We have verified, by simulation, that the corresponding state is an attractor. We emphasize that this sandpile shape is quite distinct from that generated by standard lattice sandpiles, and is close to the S-shaped sandpile observed in rotating cylinder experiments [3].

This description of the critical sandpile leads to the assertion that our model is one in which reorganization of grains predominates over surface flow. From (1) it is clear that any value of steady slope which differs from S_0 would lead to a linear growth in the first term and is therefore unstable. Thus stability enforces solutions where the average slope, for

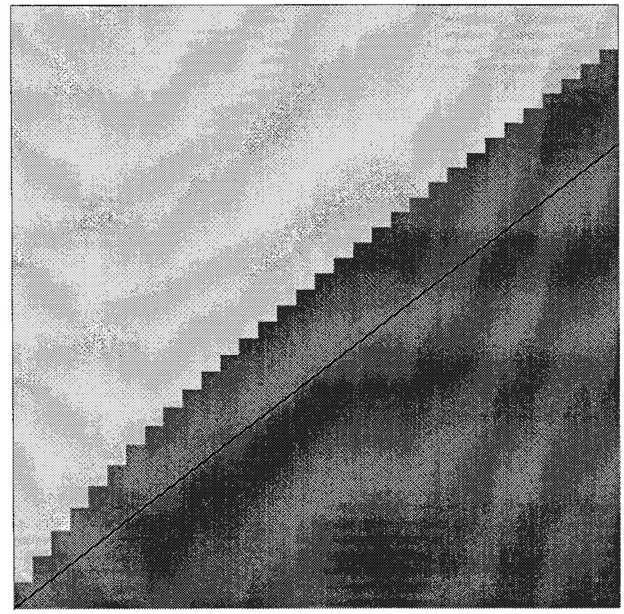

FIG. 2. The shape of a critical CML model sandpile with $L = 32$ and $\Delta = 0.95$. The line indicates the ''ideal'' column heights.

$i \gg 1$, is S_0. For a truly critical pile the second term in (1) is identically zero for $i \gg 1$ so that, except in the small i region, the threshold force that drives relaxation arises solely from the compressive component. The same reasoning is true on average for model sandpiles near criticality and makes for threshold forces which are predominantly compressive (although surface flow events arising from local slope inhomogeneities also exist, particularly near the bottom of the sandpile). This predominance of the compressive term then leads to column height changes that are typically $\sim f_{\rm th}\Delta\,\delta f$ and, in the parameter range under consideration, are small compared to the ''column grain size'' $S_0 a$ (the average step size in a lattice slope with gradient S_0 and column width a). In other words, *typical events are likely to be due to internal rearrangements generating volume changes that are small fractions of ''grain sizes''*, and they can be visualized as the slow rearrangement of grains within their clusters; this is in contrast to the surface flow events in standard lattice models [1] where entire grains flow down the surface independently of their clusters [2]. This preponderance of reorganizational events over large surface avalanches is consistent with the dynamics of sand in a slowly rotating cylinder [3,4].

The steady state response of the driven sandpile may be represented as a sequence of events each of which corresponds to a set of column height changes. Each avalanche is considered to be instantaneous so that the temporal separation of consecutive events is defined by the driving force (7). We choose a timescale in which the first column has unit growth rate and begin each simulation at $t = 0$ with a sandpile containing columns which have small and random deviations from their natural heights; also we set $a = S_0 = 1$ to fix the arbitrary horizontal and vertical length scales and we fix $f_{\rm th} = 1$ to define units of ''force.'' The dynamics of events do not depend explicitly on these choices.

In Fig. 3 we plot the distribution function per unit time and length $R[\ln(M)]$, against $\ln(M)$ for sandpiles with size

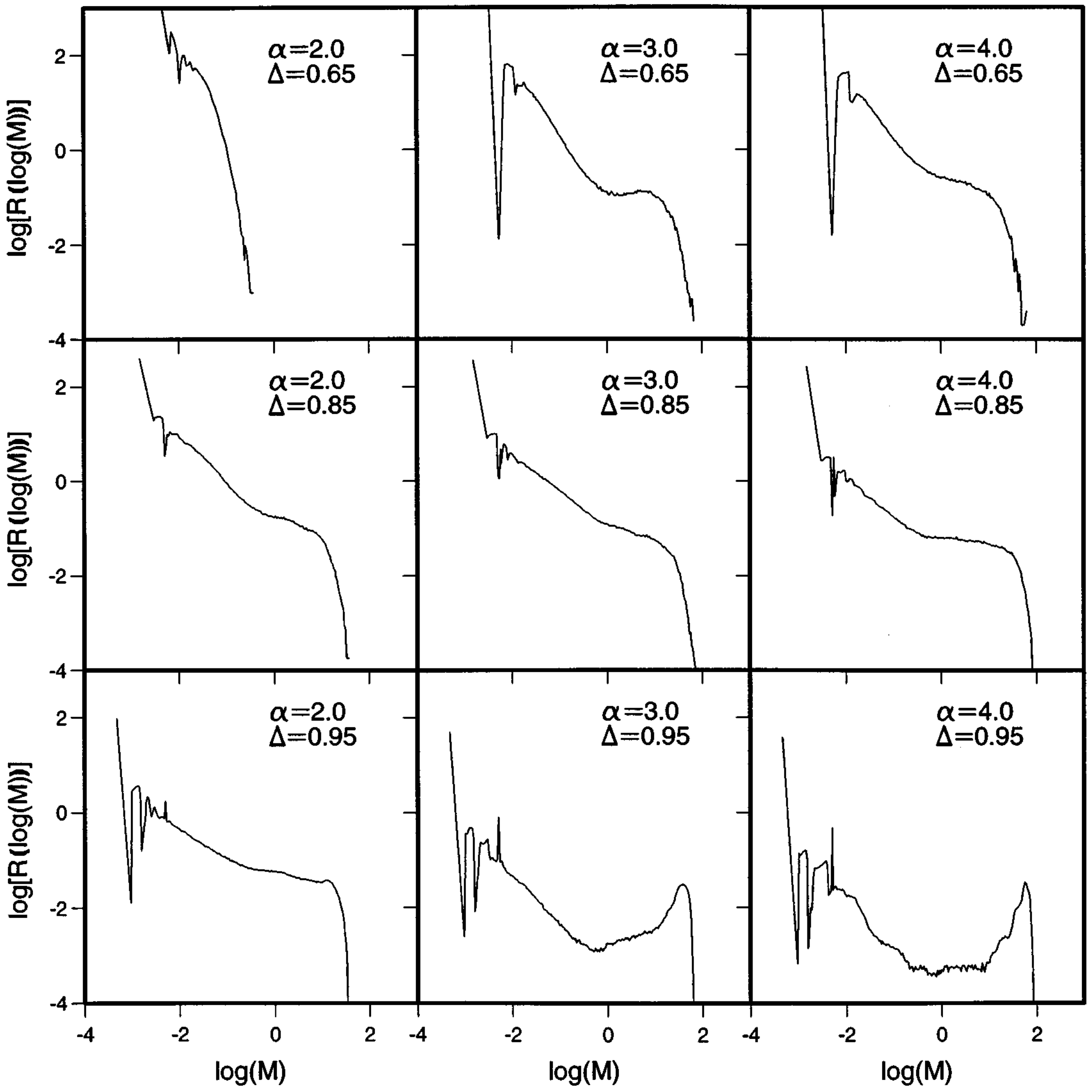

FIG. 3. A logarithmic plot of the distribution function of event sizes, $R(\mathrm{Log}(M))$, for 10^7 consecutive events in a CML model sandpile with $L=512$ and parameter values $\delta f=0.01$, $\alpha=2,3,4$ and $\Delta=0.6,0.85,0.95$. (Logarithms base 10.)

$L=512$ and parameter values $\delta f=0.01$, $\alpha=2,3,4$, and $\Delta=0.6,0.85,0.95$. We note in particular the small value of δf, and mention that our results are qualitatively unaffected by choosing δf in the range $0.001<\delta f<0.1$; given its interpretation in terms of the smallest event size, this reflects our choice of the quasistatic regime, where small cooperative internal rearrangements predominate over large single-particle motions. The distribution functions in Fig. 3 indicate a scaling behavior in the region of small magnitude events and, for larger magnitudes, frequencies that are larger than would be expected from extension of the same power law. The phase diagram in the $\Delta-\alpha$ plane indicates qualitatively distinct behavior for low-inertia, strongly consolidating (low α and Δ) systems where the magnitude distribution function has a single peak, and high-inertia, weakly consolidating (high α and Δ) systems for which the magnitude distribution has a clearly distinct second peak.

These results are in accord with the Nakanishi earthquake automaton [15]; however, their interpretation in the context of our sandpile model is quite novel and distinct (see below). In Fig. 4 we show the relative column heights, z_i, plotted against the distance of the column from the axis of rotation. The solid line denotes the configuration before, and the dotted line that after, a large avalanche; we note that a section of the sandpile has ''slipped'' quite considerably during the event.

The rotation of the sandpile causes a uniform increase of the local slopes and a preferential increase of absolute column heights in the upper region of the sandpile. The sandpile is thus driven towards its critical shape where relaxation events are triggered locally. These events will be localized (''small'') or cooperative (''large'') depending on α and Δ. For strongly consolidating systems with small amplification α, a great deal of excess volume is lost via consolidation, and the effect of surface granular flow is small; in these circumstances, the propagation and buildup of an instability is unlikely so that events are, in general, localized, uncorrelated and hence small. This leads to the appearance of the single peak in the distributions in the upper left of Fig. 3. Alternatively, for weakly consolidating systems with large amplification, the surface flow is large and large amounts of excess volume are not lost via consolidation; this situation favors

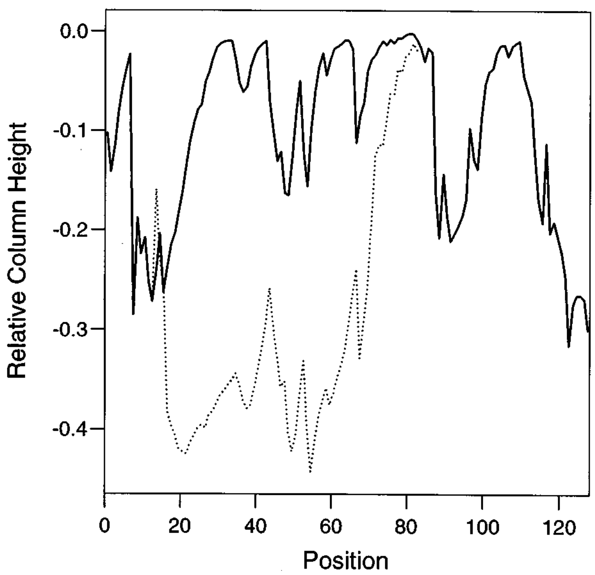

FIG. 4. A plot of column heights, relative to their critical heights, for a CML model sandpile with $L=128$ and parameter values $\delta f=0.01$, $\alpha=3$, and $\Delta=0.85$. The full (dotted) line shows the configuration before (after) a large event.

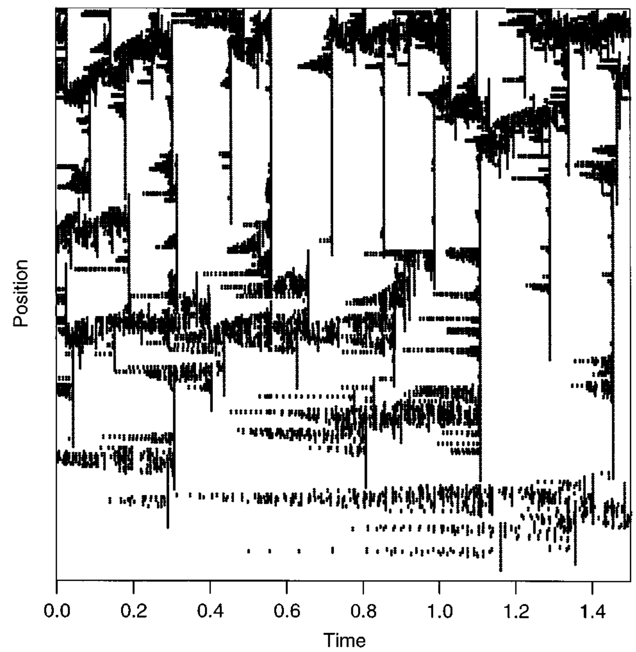

FIG. 5. A plot showing the locations of relaxation events (changes in column heights), that occur during an interval of length 1.5 which begins at $t=10^4$, for a CML model sandpile with $L=256$ and parameter values $\delta f=0.01$, $\alpha=3$, and $\Delta=0.85$.

the cooperative transmission and enhancement of events, leading to the appearance of large avalanches which are manifested by the appearance of a second peak in the distributions at the lower right of Fig. 3. In principle, these large avalanches would be halted by strong configurational inhomogeneities such as a ''dip'' on the surface, where the local driving force (1) is far below threshold; our simulations show that such configurations are rare in sandpiles that are close to criticality, and this leads to the appearance of the special scale for large avalanches.

Figure 5 shows a time series of avalanche locations that occur for a model sandpile in the two-peak region. The large events are almost periodic and each is preceded by many small precursor events; this is in accord with previous work [5,17] on sandpiles as well as earthquakes [15]. In addition it is apparent that large avalanches tend to occur repeatedly at or around the same regions of the sandpile, whose location changes only very slowly compared to the interval between the large avalanches; these correlations in both the positions and the times of large events are often referred to as ''memory'' [2–4].

We have previously given a qualitative explanation for memory in sandpiles [17] which, in the context of our present model, becomes more quantitative. Thus because the relaxation function (4) is a smooth function of the excess force f_i-f_{th} (which depends on the configuration of the pile), the propagation of a large event across the sandpile causes a smoothing of *small* configurational inhomogeneities. In turn this reduces the probability that a new event will be initiated in the same region of the sandpile, until the whole region is again driven towards its critical configuration. In contrast large configurational inhomogeneities, such as *large* dips or surface voids, which are able to halt the progress of large avalanches, remain as significant features (often slightly weakened and displaced) in the sandpile configuration following a large event; these can then have an effect on the spatial extent of subsequent events. Thus, for those regions of phase space (α and Δ large) where granular inertia plays a large role in amplifying avalanches, and where consolidation is not effective, our model shows the existence of quasiperiodic large events which repeatedly disrupt the same regions of the system, thus manifesting configurational memory. On the other hand, when the inertial effect is weak and when the void space is honed down via consolidation (α and Δ small), the predominant effect is that of small uncorrelated events which do not leave a persistent mark on the sandpile configuration so that no memory effects are observed. For moderate values of α and Δ, both small and large events will be seen (Fig. 5); note also that for the specific case of the rotated sandpile, these large events occur predominantly towards the top of the pile, where structural reorganization brought on by rotation is most effective.

The shape of the critical sandpile, which we discussed earlier, leads to another interesting feature, namely, an intrinsic size dependence. As mentioned before, the shape is characterized by (a) the length of the decreasing slope region and (b) the constant deviation of column heights from their ideal values in the steady slope region which follows (Fig. 2). The length $\mathbb{L}$ of the increasing slope region at small i has a finite extent given by

$$\mathbb{L}\sim(1+\sqrt{1-\Delta^2})/(1-\Delta+\sqrt{1-\Delta^2}) \tag{9}$$

and this can be made an arbitrary fraction of the sandpile by an appropriate choice of system size. An intrinsic size dependence resulting from the physics of competition between the surface and bulk relaxation processes is of interest because it has been observed in sandpile experiments [9,18]; however, we will defer the full effects of this to future work, and for the present limit our results to sandpiles whose length $L\gg\mathbb{L}$.

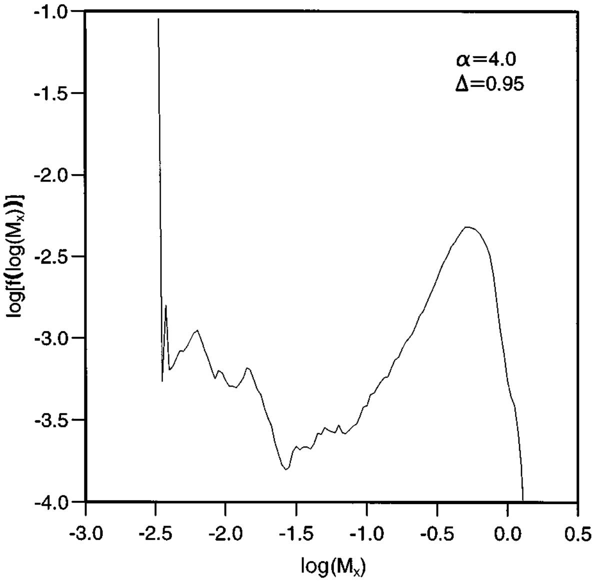

FIG. 6. A logarithmic plot of the distribution function of exit mass sizes, $f(\mathrm{Log}(m_x))$, for a sandpile of size $L=128$ with $\delta f=0.01$, $\alpha=4$ and $\Delta=0.95$ for 10^7 consecutive events. The exit mass m_x is the sum of height increments, $\delta z'$, that topple from the first column during an event. (Logarithms base 10.)

As mentioned in the introductory section, we would expect few events to result in mass exiting the pile, as our model is one in which internal volume reorganizations dominate surface flow. Thus, mass will exit a pile either via the propagation of large events (which occur for α and Δ large) or if surface flow is significant (typically events initiated in the increasing slope region $\mathbb{L}$). Figure 6 shows the logarithm of the exit mass size distribution function, $f(\ln(m_x))$, for a sandpile of size $L=128$ with $\delta f=0.01$, $\alpha=4$, and $\Delta=0.95$. While the absolute magnitudes of the event sizes are suppressed in comparison to Fig. 3, we see the two-peak behavior consistent with the corresponding event size distribution function. The large second peak indicates that a significant proportion of the exit mass is due to large events referred to above; also, we have checked that this is the only part of the distribution that survives for larger system sizes, in agreement with the length dependence above. Finally, we mention that the two-peak behavior obtained for the exit mass distribution is in accord with behavior we have observed previously in a cellular automaton model of a reorganizing sandpile [5], thus confirming our conclusion [3] that the presence of reorganization as a ''second'' mechanism of relaxation causes the breakdown of scale invariance observed in simpler models [1].

Sandpile driven by random deposition

The perturbation more usually encountered in sandpiles is random deposition. We may replace the organized addition (7) with the random sequential addition of height elements, $z_i^+=z_g$, onto columns $i\in 1,L$. When the added elements are small compared to the minimum event size, so that $z_g \ll f_{\mathrm{th}}\delta f$, random addition is statistically equivalent to uniform addition, which was the case considered in the context of earthquakes [15]. The distribution of event sizes, shown by the full lines (corresponding to $z_g=0.01$) in Fig. 7, is then not markedly distinct from that shown in Fig. 3 for the rotational driving force and both are similar to the distributions presented in [15]. In most of the parameter ranges we con-

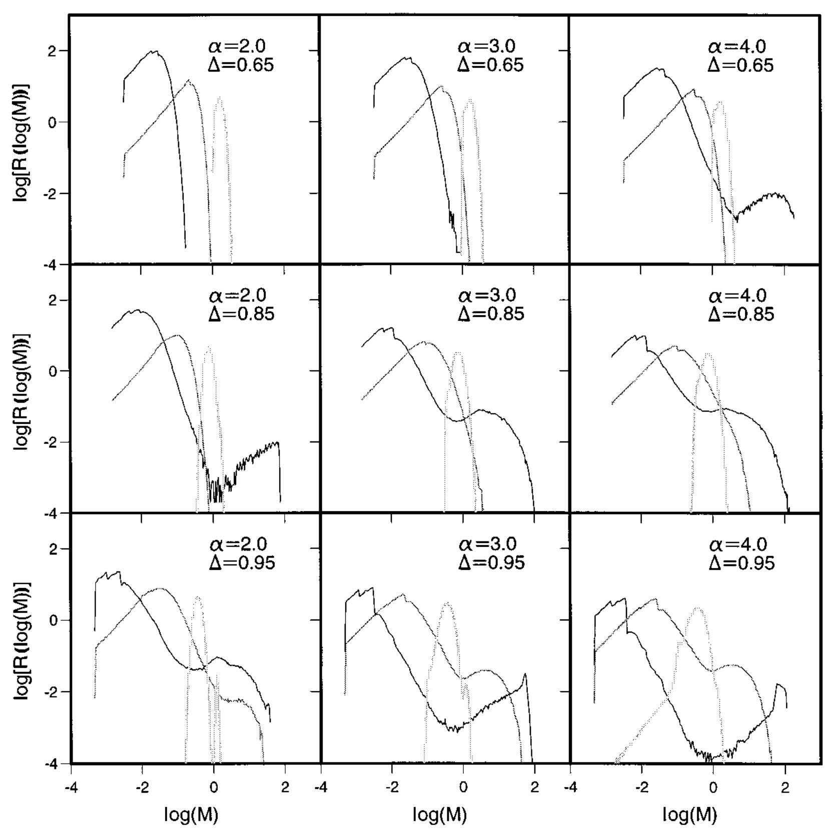

FIG. 7. A logarithmic plot of the distribution function of event sizes, $R(\mathrm{Log}(M))$ (full lines), for 10^7 consecutive events in a randomly driven CML model sandpile with $L=512$ and parameter values $\delta f=0.01$, $\alpha=2,3,4$, $\Delta=0.6,0.85,0.95$, and $z_g=0.01$. Faint lines show the corresponding distribution functions for $z_g=0.1$ and 1.0. (Logarithms base 10.)

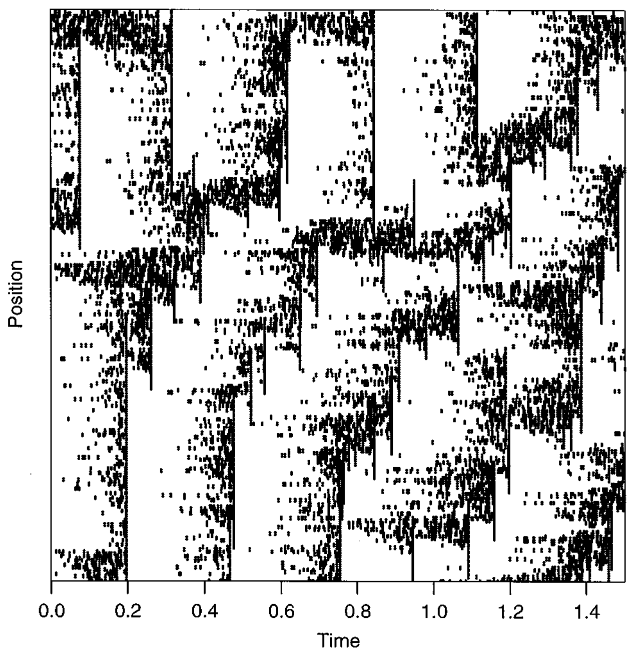

FIG. 8. A plot showing the locations of relaxation events (changes in column heights), that occur during an interval of length 1.5 which begins at $t=10^3$, for a randomly driven CML model sandpile with $L=256$ and parameter values $\delta f=0.01$, $\alpha=3$, $\Delta=0.85$, and $z_g=0.01$.

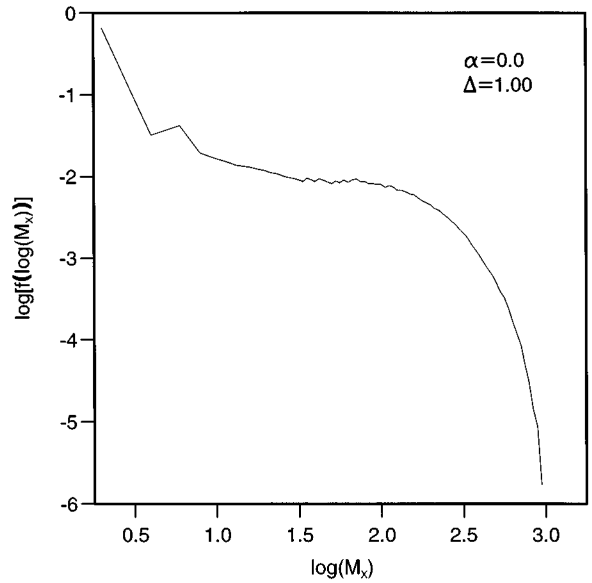

FIG. 9. A logarithmic plot of the distribution function of exit mass sizes, $f(\mathrm{Log}(m_x))$, for 10^7 consecutive events in a randomly driven CML model sandpile with $L=512$ and parameter values $\delta f=4$, $\Delta=1$, and $z_g=1$. (Logarithms base 10.)

sider, the event size distribution functions are size independent, indicating that intrinsic properties of the sandpile are responsible for their dominant features, which include a second peak representing a preferred scale for large avalanches. Figure 8 shows the corresponding time series of event locations—note that random driving leads to events which are relatively evenly spread over the sandpile and to repeated large events, each preceded by their precursor small events. Note also that after the passage of a large event and/or catastrophe over a region, there is an interval before events are generated in response to the deposition; this underlines our picture referred to earlier, whereby large events leave their signatures on the landscape, in the form, for instance of dips. These configurational sinks are associated in our model with forces well below threshold, so that grains deposited on them will, for a while, not cause any relaxation events until the appropriate thresholds are reached.

If we now increase the size of the incoming height elements so that they are larger than the minimum event size but are still small compared to the column grain size (i.e., $f_{\mathrm{th}}\delta f<z_g<aS_0$), there are two direct consequences. First, the driving force leads to local column height fluctuations $\sim z_g$, so that the surface is no longer smooth; these height fluctuations play the role of additional random barriers which impede the growth of avalanches, thus reducing the probability for large, extended events. Second, given that the added height elements are much larger than the minimum event size, their ability to generate small events is also reduced; the number of small events therefore also decreases. The size distributions for this case consequently have a domed shape with apparently two scaling regions. This case is illustrated by one of the fainter lines in Fig. 7 (corresponding to $z_g=0.1$). Note that for large α and Δ, the large events, being more persistent, are able to overcome the configurational barriers ($\sim z_g$ fluctuations in column heights) referred to above and we still see a second peak indicating the continued presence of a preferred avalanche size.

As the perturbation strength becomes even stronger, $z_g>aS_0$ so that column height fluctuations are comparable with the column grain size, there are frequent dips on the landscape, which can act as configurational traps for large events. All correlations between events begin to be destroyed and relaxation takes place locally giving a narrow range of event sizes determined only by the size of the deposited grains. This situation is illustrated by the second faint line in Fig. 7, which corresponds to $z_g=1.0$.

Finally, and for completeness, we link up with the familiar scaling behavior of lattice sandpiles [1]; as mentioned before, scale invariance is recovered when the driving force is proportional to slope differences alone and no longer contains the second mechanism of compression and/or reorganization, so that $k_1=0$ and $\Delta=1$. In order, more specifically, to match up with the local and limited $n_f=2$ model of Kadanoff *et al.* [16], we start with the randomly driven model and (a) set $k_1=0$ and $\Delta=1$ in Eq. (1), (b) set $z_g=1$, (c) choose a relaxation function $f_i-f_i'=f_{\mathrm{th}}\delta f$; this is a constant independent of f_i for $f_i>f_{\mathrm{th}}$, and in particular contains no amplification; and (d) set $\delta f=4.0$, so that each threshold force causes a minimum of two ''grains'' to fall onto the next column at every event, so that $n_f=2$.

This special case of our CML model is then identical with the scale-invariant model of Kadanoff *et al.* [16]. Figure 9 shows the smooth (scaling) exit mass size distribution in the limit of no dissipation and the corresponding spatial distribution of scaling events is shown in Fig. 10. Uncorrelated events are observed over many sizes indicating a return to scale invariance.

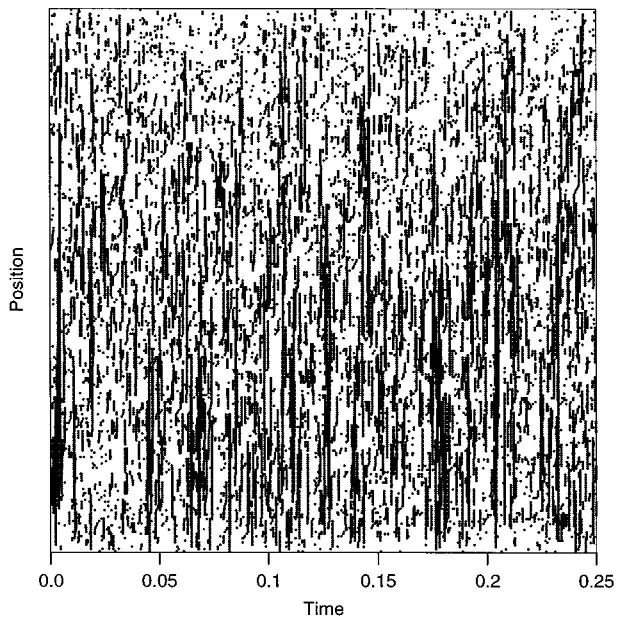

FIG. 10. A plot showing the locations of relaxation events (changes in column heights), that occur during an interval of length 0.25 which begins at $t=10^3$, for a randomly driven CML model sandpile with $L=256$ and parameter values $\delta f=4$, $\Delta=1$, and $z_g=1$.

DISCUSSION

Before discussing our main results, we review the motivation behind the work presented in this paper. The earliest cellular automaton models of sandpiles [1,16], which were put forward to illustrate self-organized criticality (SOC), relied on simple pictures of grains flowing down sloping surfaces. While they were entirely adequate in their aim, and act as paradigms of SOC, their predictions were at variance with experiments on real sandpiles [8,9]; this anomaly stimulated a great deal of work on the dynamics of granular systems [2,3]. The importance of a coupling between freely flowing surface grains and relatively immobile clusters in the bulk material soon became apparent; this idea was put forward in a model [10] encapsulating the competition between independent-particle and collective dynamics in a sandpile subjected to external perturbation, where mobile grains and clusters were respectively responsible for the two dynamical modes. A body of work proceeded to examine the relative effects of the two relaxation mechanisms on the material properties of the system [4,5,12,13]; this yielded results in agreement with experiment, thus validating the idea of grain-cluster coupling on which they were based.

Here we have provided a decorated lattice model to represent grain and cluster couplings in a sandpile. The coupled map lattice model has two important parameters; α (amplification) which determines the strength of surface flow or grain inertia effects, and Δ, which is related to internal reorganization such that $1-\Delta$ represents dissipation.

Our main result is that for large α and Δ, there is a preferred size for large avalanches, which is manifested as a second peak in the distribution of event sizes (Fig. 3). In terms of a simple picture this is because, for large α and Δ, grains have enough inertia to speed past available traps, and

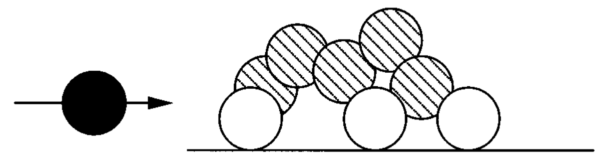

FIG. 11. A schematic diagram illustrating the mechanism for large-avalanche formation. When Δ is large, there is a great deal of undissipated volume in the cluster, resulting in the upper (shaded) grains being unstable to small perturbations. When α is large, the black grain hitting the cluster has large inertia so that a large avalanche results when it dislodges the shaded grains.

there is consequently a large amount of unrelaxed excess volume on the surface. This excess volume can be visualized, for instance, as a precariously balanced cluster (Fig. 11); the oncoming (dark) grain will knock off the shaded grains when it hits them, unleashing a large avalanche. For small α and Δ we see, by contrast, mainly small events leading to a single peak in Fig. 3; we visualize this by imagining slowly moving grains (low inertia) drifting down the surface, locking into voids and dissipating excess volume efficiently. This qualitative picture also indicates that initiated avalanches will be terminated relatively rapidly, leading to many small events.

The large avalanches mentioned above are quasiperiodic and tend to occur repeatedly around the same regions of the surface (Fig. 5), providing an important representation of configurational memory. In terms of the simple picture above this is because regions of the sandpile which look like Fig. 11 are wiped clean by the effect of the large avalanche, so that further deposition or rotation has no effect for a while. However, the effect of large α and Δ mean that once again, unrelaxed volume will be created around the same region after high-inertia grains flow down the surface; this will be the case after a number of small events have occurred (Fig. 5). These spatial and temporal correlations result in a quasiperiodic repetition of large avalanches around the same regions of the sandpile, resulting in configurational memory [17].

We have examined the response of the CML model to random deposition, with particular reference to the size z_g of the deposited grains. Three distinct regimes are observed:

(i) When z_g is of the order of a ''minimum event,'' i.e., it is comparable to the smallest fractional change in volume caused by a reorganizing grain, the response is similar to that of rotation (Fig. 7).

(ii) When z_g is intermediate between the minimum event size and the typical column grain size of the sandpile, reorganizations of grains corresponding to the smallest volume changes are ruled out; on the other hand, there are moderately sized barriers ($\sim z_g$) across the landscape impeding the progress of large events. The appropriate size distribution in Fig. 7 has, consequently, a shape which lacks the extreme small and large events of the previous case.

(ii) When z_g is larger than the column grain size of the sandpile large configurational barriers are generated by deposition and these act as traps for large events. Correlations between events are destroyed, leading to a narrow distribution of event sizes corresponding to local responses to the deposition.

The central analogy between our work and work on spring-block models of earthquakes [14,15] is *that small events build up configurational stress (in the sandpile context this is via landscapes which look like Fig. 11) which then leads, quasiperiodically, to the large events able to release the stress*. While we have drawn this analogy in previous work [4,17], the work presented here is a quantification of this analogy with an explicit model. Thus, while our model is necessarily similar to earthquake models in some respects, such as the choice of Δ governing the force redistribution, our interpretation in terms of fluctuating column heights (1), (3) is actually quite distinct.

The relaxation process (6) is symmetric and is, in that sense, similar to that presented in [15]. However the CML sandpile model has a preferred direction of flow (down the slope) created by the boundary conditions so that, on average, $z_i > z_{i-1}$ and predominantly this leads to a situation for which $f_{i-1} > f_{i+1}$. Thus, even though the relaxation process in (6) is symmetric, the propagation of relaxation events is biased in the direction down the slope. The effect of this asymmetry has far reaching consequences [19], one of which is that it is possible to solve a boundary value problem, which cannot be solved in the spring-block model [15], leading to the explicit shape of the critical pile presented in (8). Another crucially important consequence of this difference is that in the sandpile model, most of the flow comes from the compression term, whereas this is not the case for the earthquake model [15]. This feature, with a change of driving force, enables us to model the experimentally important situation of a sandpile in a slowly rotating cylinder.

We note that the limit $\Delta=1$ is a special case; this corresponds to the situation with no reorganization, where $k_1=0$, and describes a sandpile which is constantly at an ideal density. The granular driving force no longer has a compressive component and, as for standard sandpile models [1], depends only on height differences. The approach to this limit is also of interest, involving a discontinuous transition to a regime in which the critical sandpile has a constant slope (S_0+f_{th}). The neighborhood of the limit $\Delta\sim 1$, is a region of very weak dissipation, and, as has been seen in other deterministic nonlinear dynamical systems [20], could well be characterized by complex periodic motion at large times; this has recently been argued to be especially relevant to models with periodic boundaries [21]. We have investigated this issue and find that, for the regions of parameter space explored here, we do not see periodic features in sequences containing up to 5×10^7 events.

To conclude, we have designed a coupled-map lattice model of a sandpile which includes surface flow and internal reorganization as the two principal mechanisms of relaxation, focusing on the situation where reorganization dominates flow. We have, by an appropriate choice of driving forces, examined its response to rotation, as well as to deposition. In the former case, we have presented the event size distribution corresponding to different regions of parameter space and explained our results in terms of the inertia of the flowing grains and the reorganization of clusters. In the latter case, we have analyzed the event size distribution function in terms of the size of the deposited grains. We hope that these results will stimulate new investigations, both experimental and theoretical, which include an explicit examination of the couplings between internal and surface degrees of freedom for flowing sandpiles.

APPENDIX: REEXPRESSION OF THE COUPLED MAP

For the relaxation event summarized in Fig. 1, with $i\neq 1,L$, substitution of (3) into (2) gives

$$f'_{i-1}=f_{i-1}+(1+k_1/k_2)\,\delta z',$$

$$f'_i=f_i-(1+k_1/k_2)\,\delta z-\delta z',$$

and

$$f'_{i+1}=f_{i+1}+\delta z. \tag{A1}$$

For a symmetric process, where relaxed force is distributed equally in both directions,

$$f'_{i-1}-f_{i-1}=f'_{i+1}-f_{i+1}$$

so that

$$\delta z'=\delta z/(1+k_1/k_2)$$

and

$$f_i-f'_i=(1+k_1/k_2)\,\delta z+\delta z'=2\,\delta z/\Delta, \tag{A2}$$

where

$$\Delta=2(1+k_1/k_2)/[1+(1+k_1/k_2)^2].$$

The driving force ''lost'' due to a relaxation event in which $f_i\to f'_i$ is

$$(f_i-f'_i)-(f'_{i+1}-f_{i+1})-(f'_{i-1}-f_{i-1})=2\,\delta z/\Delta-2\,\delta z$$

$$=(1-\Delta)(f_i-f'_i)$$

and, therefore, $(1-\Delta)$ is a dissipation coefficient.

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